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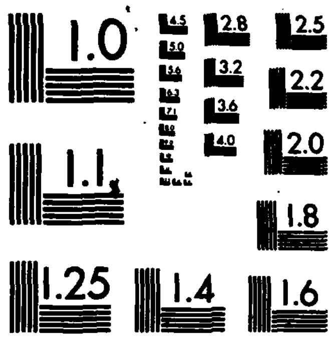

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MINIMAX METHODS FOR INDEFINITE FUNCTIONALS

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ABSTRACT

This paper contains the written version of a series of lectures presented by the author at the American Mathematical Society Summer Institute on Nonlinear Functional Analysis and Nonlinear Differential Equations. These lectures are an introduction to minimax techniques for finding critical points of functionals, especially functionals possessing symmetries. Applications are made to semilinear elliptic partial differential equations and Hamiltonian systems of ordinary differential equations.

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SIGNIFICANCE AND EXPLANATION

This paper contains an introduction to some of the ideas and methods used in finding critical points of real valued functionals by minimax arguments. The emphasis is on obtaining multiple critical points of functionals possessing symmetries. Applications are given to semilinear elliptic boundary value problems and Hamiltonian systems of ordinary differential equations.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

MINIMAX METHODS FOR INDEFINITE FUNCTIONALS

Paul H. Rabinowitz

§1. Introduction

During the past few years, there has been a considerable amount of research on obtaining critical points of indefinite functionals by means of minimax methods. The goal of these lectures is to describe some of the ideas and methods that are used in this field especially for problems involving symmetries. In the presence of symmetries one generally hopes to obtain multiple critical points.

To begin, by a functional we simply mean a mapping $I : E \to \mathbb{R}$ where E is a real Banach space. The functional I will generally be assumed to be continuously differentiable, i.e. $I \in C^1(E,\mathbb{R})$. The Frechet derivative of I at $u \in E$ acting on $\phi \in E$ is denoted by $I'(u)\phi$. A critical point of I is a point $u \in E$ at which $I'(u) = 0$, i.e. $I'(u)\phi = 0$ for all $\phi \in E$. The value of I at a critical point is called a critical value of I. In applications to differential equations, critical points of I correspond to weak solutions of the equation. Thus critical point theory serves as a useful tool for obtaining existence results for differential equations.

What are indefinite functionals? We illustrate with several examples:

Example 1.1 : Boundary value problems for semilinear elliptic partial differential equations

Consider the equation

$$(1.2)\qquad \begin{aligned} -\Delta u &= p(x,u) \quad , \quad x \in \Omega \; ; \\ u &= 0 \quad , \quad x \in \partial\Omega \end{aligned}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary. Under appropriate growth and mild smoothness conditions on p, solutions of (1.2) are critical points of

$$(1.3)\qquad I(u) = \int_\Omega \left(\frac{1}{2} |\nabla u|^2 - P(x,u)\right)dx$$

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on $E = W_0^{1,2}(\Omega)$. In (1.3), $P(x,\xi)$ is the primative of $p(x,\xi)$, i.e.

$$P(x,\xi) = \int_0^\xi p(x,t)dt \quad .$$

For an interesting class of P's, e.g. $P(x,\xi) = |\xi|^{s+1}$ where $1 < s < \frac{n+2}{n-2}$ if $n > 2$ and $s < \infty$ if $n = 2$, $I(u)$ is not bounded from above or below on E even modulo subspaces of finite dimension or codimension. Thus I is an indefinite functional.

<u>**Example 1.4 : Periodic solutions of second order Hamiltonian systems**</u>

Consider the system of ordinary differential equations:

$$\ddot{q} + V_q(q) = 0 \tag{1.5}$$

where $q \in \mathbb{R}^n$, $V \in C^1(\mathbb{R}^n,\mathbb{R})$, and $\ddot{q} \equiv \frac{d^2q}{dt^2}$. More generally, V could depend on t in a time periodic fashion. A T periodic solution of (1.5) is a critical point of

$$I(q) = \int_0^T (\tfrac{1}{2}\,|\dot{q}|^2 - V(q))dt \tag{1.6}$$

for q in an appropriate Hilbert space of T periodic functions. Once again for a large class of potential energy terms V, $I(q)$ is an indefinite functional.

<u>**Example 1.7 : Periodic solutions of general Hamiltonian systems**</u>

A general (unforced) Hamiltonian system has the form

$$\dot{z} = J\,H_z(z) \quad , \quad J = \begin{pmatrix} 0 & -id \\ id & 0 \end{pmatrix} \tag{1.8}$$

where $z = (p,q)$, $p, q \in \mathbb{R}^n$, and $H \in C^1(\mathbb{R}^{2n},\mathbb{R})$. One of the important properties of such systems is that if $z(t)$ is a solution of (1.8), then the "energy" $H(z(t))$ is independent of t. Two questions that have been studied for (1.8) are: (a) the existence of periodic solutions having a prescribed energy, e.g. $H(z(t)) = 1$; (b) the existence of periodic solutions having a prescribed period T. For (a), the period is a priori unknown so it is convenient to make a change of time scale so that the period becomes 2π and (1.8) becomes

$$\dot{z} = \lambda\, J\, H_z(z) \tag{1.9}$$

where we now seek $\lambda \neq 0$ and a 2π periodic function $z(t)$ such that $H(z(t)) = 1$. The variational formulation of this problem is: Find critical points of the so-called action integral

$$(1.10) \qquad A(z) \equiv \int_0^{2\pi} p \cdot \dot{q}\, dt$$

subject to the constraint

$$(1.11) \qquad \frac{1}{2\pi} \int_0^{2\pi} H(z)dt = 1$$

and z (and H) in an appropriate class of functions. The constraint (1.11) leads to a Lagrange multiplier in the corresponding Euler equation which is the unknown λ in (1.9). Moreover if λ, z satisfy (1.9),

$$(1.12) \qquad H(z(t)) \equiv \text{constant}$$

so (1.11) implies $z(t)$ lies on $H^{-1}(1)$.

Problem (b) corresponds to critical points of

$$(1.13) \qquad I(z) = A(z) - \lambda \int_0^{2\pi} H(z)dt \quad .$$

Once again it is not difficult to see that the functionals (1.13) and (1.10) subject to (1.11) are indefinite for a large class of Hamiltonians.

<u>Example 1.14</u> : <u>Time periodic solutions of a forced semilinear wave equation</u>

Consider

$$(1.15) \qquad \begin{cases} u_{tt} - u_{xx} + f(t,x,u) = 0 & 0 < x < \ell \\ u(0,t) = 0 = u(\ell,t) \end{cases}$$

where f is T periodic in t and we seek a solution which is also T periodic in t. The corresponding functional is

$$(1.16) \qquad I(u) = \int_0^T \int_0^\ell [\frac{1}{2}(u_t^2 - u_x^2) - F(x,t,u)]dxdt$$

which is indefinite.

Minimax methods will be used to treat such indefinite functionals. These methods characterise a critical value, c, of I as a minimax of I over an appropriate class of sets $\mathcal{K}$:

$$(1.17) \qquad c = \inf_{B \in \mathcal{K}} \sup_{u \in B} I(u) \quad .$$

As a simple example of such a result, consider the so-called <u>Mountain Pass Theorem</u>:

<u>Theorem 1.18</u> [1] : Let E be a real Banach space and suppose $I \in C^1(E,\mathbb{R})$ satisfies the Palais-Smale condition. Further assume $I(0) = 0$ and I satisfies:

(I_1) There are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho} \geq \alpha$.

(I_2) There exists $e \in E \setminus B_\rho$ such that $I(e) < 0$.

Let

$$\Gamma = \{g \in C([0,1],E) \mid g(0) = 0,\ g(1) = e\} .$$

Then

$$c = \inf_{g \in \Gamma} \max_{u \in g([0,1])} I(u) \tag{1.19}$$

is a critical value of I with $c \geq \alpha$.

In the theorem, B_ρ denotes the open ball of radius ρ about 0 in E and ∂B_ρ denotes its boundary. We will digress briefly to sketch the proof of the Mountain Pass Theorem because it illustrates some of the basic ideas used in minimax arguments. First the Palais-Smale condition, (PS), must be explained. This condition states that any sequence $(u_m) \subset E$ such that $|I(u_m)|$ is bounded and $I'(u_m) \to 0$ has a convergent subsequence. Thus (PS) is a kind of compactness condition. E.g. if $K_c \equiv \{u \in E \mid I(u) = c$ and $I'(u) = 0\}$, then (PS) implies that K_c is compact. It further implies a certain uniformity that is required to prove the following (simplest version of the)

Deformation Theorem: If $I \in C^1(E,\mathbb{R})$ and satisfies (PS), $\bar{\varepsilon} > 0$, and c is not a critical value of I, then there is an $\varepsilon \in (0,\bar{\varepsilon})$ and $\eta \in C([0,1] \times E,E)$ such that

1° $\eta(1,u) = u$ if $I(u) \notin [c-\bar{\varepsilon},c+\bar{\varepsilon}]$

2° $\eta(1,A_{c+\varepsilon}) \subset A_{c-\varepsilon}$

where $A_s \equiv \{u \in E \mid I(u) \leq s\}$.

We do not have time to go into the details of the proof here. However two quick remarks suffice to illustrate the ideas involved in the simplest setting of $E = \mathbb{R}^n$ and $I \in C^2$. Consider the ordinary differential equation:

$$\frac{d\psi}{dt} = - I'(\psi) \tag{1.20}$$

with initial condition $\psi(0;u) = u$. Then

$$\frac{d}{dt} I(\psi(t;u)) = - |I'(\psi(t;u))|^2 \leq 0$$

so except at zeroes of I', I strictly decreases along orbits of (1.20). This observation together with (PS) plays a key role in establishing 2° of the Deformation Theorem.

Actually to get 2^o, we must replace the right hand side of (1.20) by a rescaled version of itself for otherwise the solution may not exist for the full interval $t \in [0,1]$. Moreover to prove 1^o, we must also multiply the right hand side of (1.20) by an appropriate localization factor. See e.g. [2] or [3] for a complete proof.

<u>Proof of the Mountain Pass Theorem</u>: Observe first that each curve $g([0,1])$ crosses ∂B_ρ and therefore

$$\max_{g([0,1])} I(u) \geq \alpha$$

by (I_1). Hence $c \geq \alpha$ by (1.19). Suppose that c is not a critical value of I. Setting $\bar{\varepsilon} = \frac{\alpha}{2}$ and invoking the Deformation Theorem, we find $\varepsilon \in (0,\bar{\varepsilon})$ and $\eta \in C([0,1] \times E, E)$ such that

(1.21) $$\eta(1,A_{c+\varepsilon}) \subset A_{c-\varepsilon}.$$

Choose $g \in \Gamma$ such that

(1.22) $$\max_{u \in g([0,1])} I(u) \leq c+\varepsilon$$

and consider $\eta(1,g(t))$. Note that $\eta(1,g(0)) = \eta(1,0) = 0$ via 1^o of the Deformation Theorem since $I(0) = 0 < \frac{\alpha}{2} \leq c-\bar{\varepsilon}$. Similarly $\eta(1,g(1)) = \eta(1,e) = e$ via (I_2) and the above argument. Therefore $\eta(1,g(t)) \in \Gamma$. But then (1.21) - (1.22) imply

(1.23) $$\max_{u \in \eta(1,g([0,1]))} I(u) \leq c-\varepsilon \quad ,$$

contrary to the definition of c. Thus c is a critical value of I and the proof is complete.

As was mentioned earlier we will be interested mainly in symmetric functionals in these lectures. A symmetric functional is one which is invariant under a group G of mappings of E into E, i.e. $I(u) = I(gu)$ for all $g \in G$. Some examples will be given next.

<u>Example 1.24</u>: In Example 1.1, suppose $P(x,\xi)$ is even in ξ. Choosing $G \equiv \{id,-id\} \simeq \mathbb{Z}_2$ where id denotes the identity map in E, I as defined in (1.3) is invariant under G.

Example 1.25: An appropriate space to work with in Example 1.7 is $E = (W^{1/2,2}(S^1))^{2n}$, the Hilbert space of 2n-tuples of 2π periodic functions which are square integrable and possess a square integrable derivative of order $\frac{1}{2}$. (See e.g. [4] for a more precise definition.) Let $G \equiv \{g_\theta \mid \theta \in [0,2\pi)$ and $g_\theta z(t) = z(t+\theta)$ for all $z \in E\}$. Thus $G \simeq \mathbb{R}/[0,2\pi) \simeq S^1$ and I as defined in (1.13) is invariant under G.

Remark 1.26. For groups G as above, the fixed point set of G, Fix G is defined as

$$\text{Fix } G \equiv \{u \in E \mid gu = u \text{ for all } g \in G\} \quad . \tag{1.27}$$

Thus in Example 1.24, Fix $G = \{0\}$ while in Example 1.25, Fix G is the set of constant functions in E which in turn can be identified with $\mathbb{R}^{2n}$. Fix G plays an important role in problems with symmetries. Whenever Fix G is nontrivial, care must be taken to avoid it; otherwise there are difficulties in trying to exploit the symmetries to obtain multiplicity results.

Example 1.28: Let I and E be as in Example 1.3 where Ω now denotes the unit ball in $\mathbb{R}^2$. Using polar coordinates, we see I is invariant under

$$G \equiv \{g_\tau \mid \tau \in [0,2\pi) \text{ and } g_\tau(u(r,\theta)) = u(r,\theta+\tau) \text{ for all } u \in E\} \simeq S^1 \quad .$$

Here Fix G consists of those $u \in E$ which are independent of θ, i.e. Fix G consists *of the set of radial functions.* To date due to the presence of this large Fix G, no one has successfully used minimax methods to tackle this problem.

The existence of symmetries can be useful in obtaining multiple critical points of a functional. The first result of this type is due to Ljusternik [5].

Theorem 1.29: Suppose $f \in C^1(\mathbb{R}^n,\mathbb{R})$ and f is even. Then $f|_{S^{n-1}}$ has at least n distinct pairs of critical points.

A more recent result is a symmetric version of the Mountain Pass Theorem.

Theorem 1.30 [6]: Let E be a real Banach space and $I \in C^1(E,\mathbb{R})$ satisfying (PS). Suppose further I is even, $I(0) = 0$, I satisfies (I_1), and

(I_2') for all finite dimensional $\tilde{E} \subset E$, there is an $R = R(\tilde{E})$ such that $I < 0$ on $\tilde{E} \setminus B_{R(\tilde{E})}$.

Then I has an unbounded sequence of critical values.

In the remainder of these lectures we will describe some of the ingredients that go into the proofs of such multiplicity statements as well as some applications to differential equations such as (1.2) and (1.8).

§2. Index theories and some multiplicity results

In order to exploit symmetries to obtain multiple critical points of a functional, several preliminaries are required. First we need a way to measure the size of symmetric sets. An index theory is an appropriate tool for this purpose and is useful in dealing with symmetric sets in other ways. For many situations, especially unconstrained problems, intersection theorems are needed to get estimates. Classes of sets with respect to which to minimax the functional must also be found. The choice of such classes has been a very ad hoc process. Lastly a symmetric version of the Deformation Theorem is needed. We will study these matters next, mainly in a $\mathbb{Z}_2$ setting.

What is an index theory? Probably the simplest one is obtained with the aid of the notion of genus introduced by Krasnoselski [7]. The equivalent form of this notion described here is due to Coffman [8]. Let E be a real Banach space and let $\mathcal{E}$ denote the family of sets $A \subset E\backslash\{0\}$ such that A is closed in E and symmetric with respect to 0, i.e. $x \in A$ implies $-x \in A$. *For* $A \in \mathcal{E}$ *the genus of* A, *denoted by* $\gamma(A)$ equals n if there is an odd map $\phi \in C(A,\mathbb{R}^n\backslash\{0\})$ and n is the smallest integer with this property. If there does not exist a finite such n, set $\gamma(A) = \infty$. Also define $\gamma(\emptyset) = 0$. Some simple examples are in order.

<u>Example 2.1</u>: Suppose $A = B\cup(-B)$ where $B \cap (-B) = \emptyset$ and B is closed. Then $\gamma(A) = 1$ since if $\phi(x) = 1$ for $x \in B$ and $\phi(x) = -1$ for $x \in (-B)$, ϕ is odd and belongs to $C(A,\mathbb{R}\{0\})$.

<u>Example 2.2</u>: If $n > 1$, A is homeomorphic to S^n by an odd mapping, $\gamma(A) > 1$ for otherwise there exists an odd $\phi \in C(A,\mathbb{R}\backslash\{0\})$. Choose $x \in A$ such that $\phi(x) > 0$. Then $\phi(-x) < 0$ and by the Intermediate Value Theorem, ϕ must vanish somewhere along any path joining x and $-x$, a contradiction.

The next result contains the main properties of genus. Below for $A \subset E$, $N_\delta(A) \equiv \{x \in E \mid \|x-A\| < \delta\}$, i.e. $N_\delta(A)$ is a uniform δ-neighborhood of A.

Proposition 2.3: Let $A, B \in \mathcal{E}$. Then

1^o **Normalization:** If $x \neq 0$, $\gamma(\{x\} \cup \{-x\}) = 1$.

2^o **Mapping property:** If there exists an odd mapping $f \in C(A,B)$, then $\gamma(A) \leq \gamma(B)$.

3^o **Subadditivity:** $\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$.

4^o **Continuity property:** If A is compact, $\gamma(A) < \infty$ and there is a $\delta > 0$ such that $\gamma(A) = \gamma(N_\delta(A))$.

Proof: 1^o is obvious. For 2^o, the result is trivial if $\gamma(B) = \infty$. Thus suppose $\gamma(B) = n < \infty$. Suppose there exists $\phi \in C(B, \mathbb{R}^n \setminus \{0\})$ with ϕ odd. Therefore $\phi \circ f$ is odd and belongs to $C(A, \mathbb{R}^n \setminus \{0\})$. Consequently $\gamma(A) \leq n = \gamma(B)$. To prove 3^o, suppose $\gamma(A) = m$, $\gamma(B) = n$, and both are finite (since again, if not, the result is trivial). Therefore there are odd functions $\phi \in C(A, \mathbb{R}^m \setminus \{0\})$, $\psi \in C(A, \mathbb{R}^n \setminus \{0\})$. Extend ϕ, ψ respectively to $\hat{\phi} \in C(E, \mathbb{R}^m)$, $\hat{\psi} \in C(E, \mathbb{R}^n)$. Replacing $\hat{\phi}, \hat{\psi}$ by their odd parts, we can assume $\hat{\phi}, \hat{\psi}$ are odd. Set $f = (\hat{\phi}, \hat{\psi})$. Then f is odd and belongs to $C(A \cup B, \mathbb{R}^{m+n} \setminus \{0\})$. Consequently $\gamma(A \cup B) \leq m+n = \gamma(A) + \gamma(B)$. Lastly to verify 4^o, for $x \in A$, let $r(x) \equiv \frac{1}{2} \|x\| = r(-x)$ and $B_{r(x)}(x) = \{u \in E \mid \|u-x\| < r(x)\}$. Set $T_x = B_{r(x)}(x) \cup B_{r(x)}(-x)$ and observe that $A \subset \bigcup_{x \in A} T_x$. By compactness, finitely many T_{x_r} cover A. Since $\gamma(T_{x_r}) = 1$, $\gamma(A) < \infty$ by 3^o. Applying 2^o with $f = \mathrm{id}$, $\gamma(A) \leq \gamma(N_\delta(A))$. Suppose $\gamma(A) = n$. Choose $\phi \in C(A, \mathbb{R}^n \setminus \{0\})$ with ϕ odd and extend ϕ to $\hat{\phi} \in C(E, \mathbb{R}^n)$ as in 3^o with $\hat{\phi}$ odd. Since A is compact, $\hat{\phi} \neq 0$ on $N_\delta(A)$ for some $\delta > 0$. Therefore $\gamma(N_\delta(A)) \leq n = \gamma(A)$.

Remark 2.4: For arguments given later it is useful to observe that if $\gamma(B) < \infty$, then $\gamma(\overline{A \setminus B}) \geq \gamma(A) - \gamma(B)$. Indeed $A \subset \overline{A \setminus B} \cup B$ so this follows from $2^o - 3^o$ of Proposition 2.3. Also observe that if $\gamma(A) > 1$, the definition of genus implies A contains infinitely many distinct pairs of points.

If E is infinite dimensional, the following result shows how to obtain sets of arbitrary genus in E.

Proposition 2.5: If $A \in \mathcal{E}$, Ω is a bounded neighborhood of 0 in $\mathbb{R}^k$, and there exists a homeomorphism $h \in C(A, \partial\Omega)$ with h odd, then $\gamma(A) = k$.

Proof: Clearly $\gamma(A) \le k$. If $\gamma(A) < k$, there exists an odd map $\phi \in C(A, \mathbb{R}^j \setminus \{0\})$ where $j < k$. The map $\phi \circ h^{-1}$ is then odd and belongs to $C(\partial\Omega, \mathbb{R}^j \setminus \{0\})$. But the existence of such a map is contrary to the Borsuk-Ulam Theorem [9]. Therefore $\gamma(A) = k$.

The next proposition is a simple example of an intersection theorem.

<u>Proposition 2.6</u>: If $\gamma(A) > k$ and V is a subspace of E of codimension k, then $V \cap A \neq \emptyset$.

Proof: Suppose $V \cap A = \emptyset$. Let P denote the projector of E onto $V^\perp$ where $V \oplus V^\perp = E$. Then P is odd and $P \in C(A, V^\perp \setminus \{0\})$. By 2° of Proposition 2.3, $\gamma(A) \le \gamma(PA)$. Projecting PA radially onto $\partial B_1 \cap V^\perp$ and using 2° of Proposition 2.3 and Proposition 2.5 yields

$$\gamma(A) \le \gamma(PA) \le \gamma(\partial B_1 \cap V^\perp) = k \quad ,$$

contrary to hypothesis.

More generally, let E be a real Banach space with a group of symmetries G on it, e.g. $\mathbb{Z}_2$, S^1, etc. Let $\mathcal{E}$ denote the set of $A \subset E \setminus \{0\}$ such that A is closed in E and invariant under G, i.e. $A \in \mathcal{E}$ and $x \in A$ implies $gx \in A$ for all $g \in G$. An <u>index theory</u> is a mapping $i : \mathcal{E} \to \mathbb{N} \cup \{\infty\}$ such that for all $A, B \in \mathcal{E}$,

1° <u>Normalization</u>: If $x \notin \operatorname{Fix} G$, then $i\left(\bigcup_{g \in G} gx\right) = 1$.

2° <u>Mapping property</u>: If $f \in C(A,B)$ and f is equivariant, i.e. $fg = gf$ for all $g \in G$, then $i(A) \le i(B)$.

3° <u>Subadditivity</u>: $i(A \cup B) \le i(A) + i(B)$.

4° <u>Continuity Property</u>: If A is compact and $A \cap \operatorname{Fix} G = \emptyset$, then $i(A) < \infty$ and there exists a $\delta > 0$ such that $i(N_\delta(A)) = i(A)$.

<u>Remark 2.7</u>: If $A \in \mathcal{E}$ and $A \cap \operatorname{Fix} G \neq \emptyset$, $i(A) = i(E)$. Indeed if $x \in A \cap \operatorname{Fix} G$, the map $f(u) = x$, $A \to \{x\}$ is continuous and equivariant so by the mapping property, $i(A) \le i(\{x\})$. But 2° with $f = id$ shows $i(\{x\}) \le i(A)$ so equality holds here. Since A can be replaced by E in this computation, $i(A) = i(E)$.

Remark 2.8: There are analogues of Proposition 2.5 for more general index theories. E.g. if $G \simeq S^1$ and i is the index theory defined in [10] or [11], $i(S^n) = n$ where S^n is a $2n - 1$ dimensional invariant sphere lying in $E \backslash \mathrm{Fix}\, G$. In particular if $E \backslash \mathrm{Fix}\, G$ is infinite dimensional and $G \simeq S^1$, combining this observation with Remark 2.7 shows $i(A) = \infty$ for any $A \in E$ such that $A \cap \mathrm{Fix}\, G \neq \emptyset$ and i is as in [10,11].

Several index theories can be found in the literature. The first one introduced was based on the notion of category due to Ljusternik and Schnirelman [12]. We have already mentioned the "geometrical" index theory provided by the notion of genus. An S^1 version of genus was given by Benci [13]. Cohomological index theories can be found in [10-11] and the references cited there.

Now we turn to the use of index theories to obtain multiplicity results for symmetric functionals. We will mainly work with genus but will sketch the use of an S^1 index theory for (1.8). Theorem 1.28 is one of the simplest multiple critical point results. Before proving it, two remarks are needed.

Remark 2.9: The version of the Deformation Theorem given earlier does not suffice for multiplicity results. Moreover when treating functionals on a manifold as in Theorem 1.28, a variant of the Deformation Theorem more suitable for such a setting is required. The following result is sufficient for Theorem 1.28. For $f : S^{n-1} \to R$, let $\hat{A}_s = \{x \in S^{n-1} \mid f(x) \le s\}$. For $c \in R$ set $\hat{K}_c = \{x \in S^{n-1} \mid f(x) = c$ and $f'(x) - (f'(x),x)x = 0\}$. Here $(\cdot,\cdot)$ denotes inner product.

Theorem 2.10: If $f \in C^1(R^n,R)$ and is even, $c \in R$, and $\mathcal{O}$ is any symmetric neighborhood of $\hat{K}_c$ in S^{n-1}, then there exists a mapping $\eta \in C([0,1] \times S^{n-1}, S^{n-1})$ and an $\varepsilon > 0$ such that

1° $\eta(t,x)$ is odd in x.

2° $\eta(1,\hat{A}_{c+\varepsilon} \backslash \mathcal{O}) \subset \hat{A}_{c-\varepsilon}$.

3° If $\hat{K}_c = \emptyset$, $\eta(1,\hat{A}_{c+\varepsilon}) \subset \hat{A}_{c-\varepsilon}$.

Remark 2.11: With $E = R^n$, set $\Gamma_j = \{A \in E \mid A \subset S^{n-1}$ and $\gamma(A) \ge j\}$, $1 \le j \le n$. Note the following four properties of the sets Γ_j:

(i) $\Gamma_j \neq \emptyset$, $1 \le j \le n$.

(ii) Monotonicity property: $\gamma_1 \supset \gamma_2 \supset \ldots \supset \gamma_n$.

(iii) Invariance property: If $\phi \in C(S^{n-1}, S^{n-1})$ and i. c id, and $A \in \gamma_j$, then $\phi(A) \in \gamma_j$.

(iv) Excision property: If $A \in \gamma_j$ and $B \in E$ with $\gamma(B) \le s < j$, then $\overline{A \setminus B} \in \gamma_{j-s}$.

Indeed property (i) follows from Proposition 2.5 with $\Omega = S^{j-1}$, $1 \le j \le n$, (ii) is trivial, (iii) is a consequence of 2^o of Propostion 2.3 and (iv) follows from Remark 2.4.

Proof of Theorem 1.28: Define

$$c_j = \inf_{A \in \gamma_j} \max_{x \in A} f(x) \quad , \quad 1 \le j \le n \quad .$$

By property (ii), $c_1 \le c_2 \le \ldots \le c_n$. We claim c_j is a critical value of $f|_{S^{n-1}}$. This fact in itself is not sufficient to prove the Theorem since possibly $c_j = \ldots = c_{j+p}$ for $p > 1$ and there is only one pair of critical points corresponding to this degenerate critical value. However we further claim if $c_j = \ldots = c_{j+p} \equiv c$, then $\gamma(\hat{K}_c) \ge p+1$. Remark 2.4 then shows there are infinitely many critical points corresponding to c. It suffices to verify the second claim since it contains the first. Suppose $\gamma(\hat{K}_c) \le p$. Then by 4^o of Proposition 2.3, there is a $\delta > 0$ such that $\gamma(N_\delta(\hat{K}_c)) \le p$ and by 2^o of Proposition 2.3, if $\hat{N} = N_\delta(\hat{K}_c) \cap S^{n-1}$, $\gamma(\hat{N}) \le p$. By Theorem 2.10 with $\mathcal{O} = \hat{N}$, there exists an $\eta \in C([0,1] \times S^{n-1}, S^{n-1})$ with $\eta(1,\cdot)$ odd and $\varepsilon > 0$ such that

(2.12) $$\eta(1, \hat{A}_{c+\varepsilon} \setminus \hat{N}) \subset \hat{A}_{c-\varepsilon} \quad .$$

Choose $A \in \gamma_{j+p}$ so that

(2.13) $$\max_A f \le c+\varepsilon \quad .$$

Then by (iv) of Remark 2.11, $\overline{A \setminus \hat{N}} \in \gamma_j$ and by (iii) of the same result, $B \equiv \eta(1, \overline{A \setminus \hat{N}}) \in \gamma_j$. Consequently by (2.12) - (2.13),

$$c = c_j \le \max_B f \le c_j - \varepsilon \quad ,$$

a contradiction.

There are generalizations of Theorem 1.28 to infinite dimensional settings due to Ljusternik [5], Browder [14], Berger [15], Amann [16], and many others. These abstract

theorems have then been applied to obtain existence of multiple solutions of nonlinear partial differential equations. Due to lack of time we will not be able to go into more detail here but turn instead to a symmetric version of the Mountain Pass Theorem. A somewhat more general result than that stated in Theorem 1.29 will be treated next.

<u>Theorem 2.14</u>: Let E be an infinite dimensional Banach space, and let $I \in C^1(E,\mathbb{R})$ be even and satisfy (PS) and $I(0) = 0$. Suppose $E = V \oplus X$ where V is finite dimensional and I satisfies

(I_1') There are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho \cap X} \geq \alpha$,

(I_2') For all finite dimensional subspaces $\hat{E} \subset E$, there exists $R(\hat{E})$ such that $I \leq 0$ on $\hat{E}\setminus B_{R(\hat{E})}$.

Then I possesses an unbounded sequence of critical values.

There is also a finite dimensional version of Theorem 2.12 which is proved in a similar but simpler fashion. The proof of Theorem 2.14 follows the same pattern as that of Theorem 1.28. First we need to supplement the statement of the Deformation Theorem with the fact that if I is even, then $\eta(1,\cdot)$ can be taken to be odd and if $\mathcal{O}$ is any symmetric neighborhood of K_c, then η can be chosen to satisfy $\eta(1,A_{c+\varepsilon}\setminus\mathcal{O}) \subset A_{c-\varepsilon}$. To continue, a class of sets, Γ_j, will be introduced. These sets possess properties like those verified for γ_j in Remark 2.11. Minimaxing I over Γ_j the produces the critical values c_j of I. Lastly one additional argument shows the c_j form an unbounded sequence.

To begin the proof, suppose V is k dimensional and $V = \text{span}\{e_1,\ldots,e_k\}$. For $m > k$, inductively choose $e_{m+1} \notin \text{span}\{e_1,\ldots,e_m\} \equiv E_m$. Set $R_m \equiv R(E_m)$ and $D_m = B_{R_m} \cap E_m$. Define

(2.15) $$G_m \equiv \{h \in C(D_m,E) \mid h \text{ is odd and } h = id \text{ on } \partial B_{R_m} \cap E_m\} .$$

Then $G_m \neq \emptyset$ since $id \in G_m$. Set

$$\Gamma_j = \{h(\overline{D_m\setminus Y}) \mid m \geq j,\ h \in G_m,\ Y \in \mathcal{E},\ \gamma(Y) \leq m-j\} .$$

<u>Proposition 2.16</u>: The sets Γ_j possess the following properties:

1° $\Gamma_j \neq \emptyset$ for all $j \in \mathbb{N}$.

2° (Monotonicity): $\Gamma_{j+1} \subset \Gamma_j$.

3° (Invariance): If $\phi \in C(E,E)$ is odd and $\phi = \mathrm{id}$ on $\partial B_{R_m} \cap E_m$ for all $m \in \mathbb{N}$, then $\phi : \Gamma_j \to \Gamma_j$ for all $j \in \mathbb{N}$.

4° (Excision): If $B \in \Gamma_j$, $Z \in \mathcal{E}$, and $\gamma(Z) \le s < j$, then $\overline{B\backslash Z} \in \Gamma_{j-s}$.

Proof: The proof is straightforward but tedious and will be omitted.

Now we define

(2.17)
$$c_j = \inf_{B \in \Gamma_j} \max_{u \in B} I(u) \quad , \quad j \in \mathbb{N} \ .$$

By 2° of Proposition 2.16, $c_{j+1} \ge c_j$. In order to show that c_j is a critical value of I, a lower bound for c_j is required. The following intersection theorem leads to such an estimate.

Proposition 2.18: If $j > k = \dim V$ and $B \in \Gamma_j$, then $B \cap X \cap \partial B_\rho \ne \emptyset$.

Proof: Let $B \in \Gamma_j$ so $B = h(\overline{D_m\backslash Y})$ where $m \ge j$ and $\gamma(Y) \le m-j$. The definition of R_m implies $I(u) \le 0$ if $u \in E_m\backslash B_{R_m}$ and $I \ge \alpha$ on $\partial B_\rho \cap X$ by (I_1'). Since $m > k$, $X \cap D_m \ne \emptyset$. Therefore $R_m > \rho$. Set $\hat{\mathcal{O}} = \{x \in D_m \mid h(x) \in B_\rho\}$ and let $\mathcal{O}$ be the component of $\hat{\mathcal{O}}$ containing 0. Since h is odd and $h = \mathrm{id}$ on $\partial B_{R_m} \cap E_m$, $\mathcal{O}$ is a symmetric bounded neighborhood of 0 in E_m. By Proposition 2.5, $\gamma(\partial\mathcal{O}) = m$. Set $W = \{x \in D_m \mid h(x) \in \partial B_\rho\}$. If $x \in \partial\mathcal{O}$, then $h(x) \in \partial B_\rho$. Therefore $\gamma(W) \ge \gamma(\partial\mathcal{O}) = m$ by 2° of Proposition 2.3. By Remark 2.4, $\gamma(\overline{W\backslash Y}) \ge m - (m-j) = j > k$. Hence $\gamma(h(\overline{W\backslash Y})) > k$ by 2° of Proposition 2.3. Consequently the definition of W and Proposition 2.6 show $\partial B_\rho \supset h(\overline{W\backslash Y}) \cap X \ne \emptyset$. But $B \supset h(\overline{W\backslash Y})$. Therefore $B \cap X \cap \partial B_\rho \ne \emptyset$.

Remark 2.19: An inspection of the above proof shows the stronger conclusion $\gamma(B \cap X \cap \partial B_\rho) \ge j-k$ holds.

Now the lower bound for c_j mentioned above can be obtained.

Corollary 2.20: If $j > k$, $c_j \ge \alpha$.

Proof: This is immediate from Proposition 2.18, (I_1'), and the definition of c_j.

The next result shows c_j is a critical value of I for $j > k$ and also gives a multiplicity statement for "degenerate" critical values.

Proposition 2.21: If $j > k$ and $c_j = c_{j+1} = \ldots = c_{j+p} \equiv c$, then $\gamma(K_c) \ge p+1$.

Proof: By Corollary 2.20, $c > \alpha > 0$. Since $I(0) = 0$, $0 \notin K_c$ and $K_c \in \mathcal{E}$. Moreover (PS) implies K_c is compact. If $\gamma(K_c) < p$, by 4° of Proposition 2.3, there is a $\delta > 0$ such that $\gamma(N_\delta(K_c)) < p$. The stronger version of the Deformation Theorem mentioned above with $\mathcal{O} = N_\delta(K_c)$ and $\bar{\varepsilon} = \frac{\alpha}{2}$ yields the existence of an $\varepsilon \in (0,\bar{\varepsilon})$ and $\eta \in C([0,1] \times E, E)$ with $\eta(1,\cdot)$ odd such that

$$\eta(1, A_{c+\varepsilon} \backslash \mathcal{O}) \subset A_{c-\varepsilon} \quad . \tag{2.22}$$

Choose $B \in \Gamma_{j+p}$ such that

$$\max_B I \leq c+\varepsilon \quad . \tag{2.23}$$

By 4° of Proposition 2.16, $Q \equiv \overline{B\backslash \mathcal{O}} \in \Gamma_j$. Moreover by 3° of Proposition 2.16, our choice of $\bar{\varepsilon}$, and 1° of the Deformation Theorem, $\eta(1,Q) \in \Gamma_j$. Therefore (2.21) - (2.22) show

$$c \leq \max_{\eta(1,Q)} I \leq c-\varepsilon \quad ,$$

a contradiction.

The final step in the proof of Theorem 2.14 is given by

Proposition 2.24: $c_j \to \infty$ as $j \to \infty$.

Proof. We use a variant of the argument of Proposition 2.21. It was observed earlier that $c_{j+1} \geq c_j$ for all $j \in \mathbb{N}$. *If the sequence* (c_j) *is bounded,* $c_j \to \bar{c} < \infty$. If $c_j = \bar{c}$ for all large j, $\gamma(K_{\bar{c}}) = \infty$ via Proposition 2.21. But (PS) implies $K_{\bar{c}}$ is compact so $\gamma(K_{\bar{c}}) < \infty$ by 4° of Proposition 2.3. Thus $\bar{c} > c_j$ for all j. Let $K = \{u \in E \mid c_{k+1} \leq I(u) \leq \bar{c}$ and $I'(u) = 0\}$. Again applying (PS) and 4° of Proposition 2.3, we see K is compact and there is a $\delta > 0$ and $q \in \mathbb{N}$ such that $\gamma(K) = q = \gamma(N_\delta(K))$. Invoking the Deformation Theorem with $\mathcal{O} = N_\delta(K)$ and $\bar{\varepsilon} = \bar{c} - c_q$ yields $\varepsilon \in (0,\bar{\varepsilon})$ and $\eta(1,\cdot) \in C([0,1] \times E,E)$ with $\eta(1,\cdot)$ odd and

$$\eta(1, A_{\bar{c}+\varepsilon} \backslash \mathcal{O}) \subset A_{\bar{c}-\varepsilon} \quad . \tag{2.25}$$

Choose $j \in \mathbb{N}$ such that $c_j > \bar{c}-\varepsilon$ and $B \in \Gamma_{j+q}$ satisfying

$$\max_B I \leq \bar{c}+\varepsilon \quad . \tag{2.26}$$

By the argument of Proposition 2.21, $\eta(1,\overline{B\backslash \mathcal{O}}) \in \Gamma_j$. Therefore

$$c_j \leq \max_{\eta(1,\overline{B\backslash \mathcal{O}})} I \leq \bar{c}-\varepsilon < c_j \quad ,$$

a contradiction.

The proof of Theorem 2.14 is now complete. The earliest version of this theorem can be found in [1] where applications were also given to a class of "superlinear" elliptic partial differential equations of the form (1.2). A more general such application will be sketched next. Consider

$$-\Delta u = p(x,u),\ x \in \Omega \quad ; \quad u = 0,\ x \in \partial\Omega \tag{2.27}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with a smooth boundary. The function p is assumed to satisfy the following conditions:

(p_1) $p \in C(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$

(p_2) $|p(x,\xi)| \le a_1 + a_2|\xi|^s$ where $1 < s < \frac{n+2}{n-2}$ and $n > 2$.

(p_3) $0 < \mu P(x,\xi) \equiv \mu \int_0^\xi p(x,\tau)d\tau \le \xi\, p(x,\xi)$ for $|\xi|$ large

(p_4) $p(x,\xi)$ is odd in ξ.

<u>Theorem 2.28</u>. If p satisfies $(p_1) - (p_4)$, (2.27) possesses an unbounded sequence of weak solutions.

<u>Remark 2.29</u>. If (p_1) is slightly strengthened, e.g. to $p(x,\xi)$ is locally Hölder continuous, then this condition together with (p_2) imply weak solutions of (2.27) are classical solutions. If $n = 1$ or 2, (p_2) can be considerably weakened.

<u>Proof of Theorem 2.28</u>: Set

$$I(u) = \int_\Omega (\frac{1}{2}\, |\nabla u|^2 - P(x,u))dx$$

for $u \in E \equiv W_0^{1,2}(\Omega)$ where as norm on E we take

$$\|u\| = (\int_\Omega |\nabla u|^2 dx)^{1/2} .$$

Since critical points of I are weak solutions of (2.27), the result is immediate if I satisfies the hypotheses of Theorem 2.14. It is clear that $I(0) = 0$ and (p_4) shows I is even. Hypotheses (p_1) and (p_2) imply that $I \in C^1(E,\mathbb{R})$ and $(p_1) - (p_3)$ imply that (PS) is satisfied. See e.g. [1] for the details here. To check (I_2'), integrating (p_3) shows there are constants $a_3, a_4 > 0$ such that

$$P(x,\xi) \geq a_3|\xi|^\mu - a_4 \tag{2.30}$$

for all $\xi \in \mathbb{R}$. Therefore

$$I(u) \leq \int_\Omega (\tfrac{1}{2} |\nabla u|^2 - a_3|u|^\mu + a_4)dx \tag{2.31}$$

for all $u \in E$. In particular for $u \in \hat{E}$, a finite dimensional subspace of E, the $|u|^\mu$ term in (2.31) dominates as $u \to \infty$ since $\mu > 2$. Thus (2.31) guarantees the existence of $R(\hat{E})$ in (I_2').

To verify (I_1'), E must be decomposed into $V \oplus X$. Choose $V = \mathrm{span}\{v_1,\ldots,v_k\}$ where k is free for the moment. The functions v_j are the eigenfunctions of

$$-\Delta v = \lambda v,\ x \in \Omega \ ; \ v = 0,\ x \in \partial\Omega \tag{2.32}$$

normalized by $\|v\| = 1$ and ordered by increasing magnitude of the correspond-ing eigenvalues. Set $X = V^\perp$, the orthogonal complement of V and consider $I|_{\partial B_\rho \cap X}$. By (p_2), for $u \in \partial B_\rho$,

$$I(u) \geq \frac{1}{2}\rho^2 - \int_\Omega (a_5|u|^{s+1} + a_6)dx \tag{2.33}$$

for some constants a_5 and a_6. By the Gagliardo-Nirenberg inequality [17],

$$\|u\|_{L^{s+1}(\Omega)} \leq a_7\|u\|^\beta \|u\|^{1-\beta}_{L^2(\Omega)} \tag{2.34}$$

where

$$\frac{1}{s+1} = \beta(\frac{1}{2} - \frac{1}{n}) + (1-\beta)\frac{1}{2} \ .$$

If $u \in X$, we have the elementary estimate

$$\lambda_{k+1}\|u\|^2_{L^2(\Omega)} \leq \|u\|^2 \tag{2.35}$$

where λ_j denotes the j^{th} eigenvalue of (2.32). Substituting (2.33) - (2.34) into (2.32) yields

$$I(u) \geq \rho^2(\frac{1}{2} - a_8\lambda_{k+1}^{-\frac{(1-\beta)(s+1)}{2}}\rho^{s-1}) - a_9 \tag{2.36}$$

for $u \in \partial B_\rho$ X. Choose $\rho = \rho(k)$ so that the term in parenthesis equals $\frac{1}{4}$, i.e.

$$\rho(k) = (\frac{1}{4a_8}\lambda_{k+1}^{\frac{(1-\beta)(s+1)}{2}})^{s-1}$$

and

$$I(u) \geq \frac{1}{4}\rho(k)^2 - a_9 \ . \tag{2.37}$$

Since $\lambda_k \to \infty$ as $k \to \infty$, $\rho(k) \to \infty$ as $k \to \infty$. Choose k such that $\rho(k)^2 > 8a_9$.

Therefore $I(u) > \frac{1}{8}\rho(k)^2 \equiv \alpha$ and (I_2') holds. The sketch of the proof of Theorem 2.28 is complete.

The multiplicity theorems presented thus far are in a Z_2 setting. Analogous results can be obtained when the symmetry group is S^1 with applications to second order and general Hamiltonian systems. In the remainder of this section we will briefly sketch some such extensions in the setting of (1.8). Thus consider

$$\dot{z} = J\,H_z(z) \quad . \tag{2.38}$$

When H grows at a "superquadratic" rate as $|z| \to \infty$, there is an analogue of Theorem 2.28 for the corresponding functional

$$I(z) = \int_0^T (p\cdot\dot{q} - H(z))dt \quad . \tag{2.39}$$

A complete treatment can be found in [18]. We will outline the result emphasizing its relationship with the previous case.

<u>Theorem 2.40</u>: If $H \in C^1(\mathbb{R}^{2n},\mathbb{R})$ and there exists $\mu > 2$ such that

(H_1) $$0 < \mu\, H(z) \le z\cdot H_z \quad \text{for all large } |z| \quad ,$$

then for each $T > 0$, (2.38) possesses a sequence of T periodic solutions which is unbounded in L^∞.

<u>Proof</u>: For convenience set $T = 2\pi$. A natural space in which to treat the first term in $I(z)$ is $E \equiv (W^{1/2,2}(S^1))^{2n}$, the Hilbert space of 2n-tuples of 2π periodic functions which possess a square integrable "derivative" of order $\frac{1}{2}$. However the second term, $\int_0^{2\pi} H(z)dt$ need not be defined on E since we have not imposed any growth restrictions on H. This creates a technical problem which one can get around by truncating H so that the new functional belongs to $C^1(E,\mathbb{R})$ and via (H_1) satisfies (PS). We will ignore this point and suppose I in (2.39) is continuously differentiable on E and satisfies (PS). See [18] for a precise treatment of the technicalities indicated above.

The space E can be decomposed into $E^+ \oplus E^0 \oplus E^-$ where $E^\pm$ are respectively the subspaces of E on which A is positive definite and negative definite and $E^0 \simeq \mathbb{R}^{2n}$ is the set of 2n-tuples of constants. Any $z \in E$ can be written as $z = z^+ + z^0 + z^- \in E^+ \oplus E^0 + E_-$. In an appropriate basis for E, which is easy to write down,

$$A(z) \equiv \int_0^{2\pi} p\cdot\dot{q}dt = \|z^+\|^2 - \|z^-\|^2 \quad .$$

The eigenvectors in E^+, E^- of A occur in pairs due to the S^1 symmetry. Ordering them by the magnitude of the corresponding eigenvalue, let

$$L_m = \text{span}\{1^{st}\ 2m \text{ eigenvectors in } E^+\} \oplus E^0 \oplus E^- .$$

Our assumption on H and the form of I shows there exists an $R_m > 0$ such that $I(u) < 0$ if $u \in L_m \setminus B_{R_m}$. Thus as in the proof of Theorem 2.14, set $D_m = B_{R_m} \cap L_m$ and

$$G_m = \{h \in C(D_m,E) | h \text{ satisfies (i) - (iii)}\}$$

where

(i) h is equivariant,

(ii) $h = id$ on $(\partial B_{R_m} \cap L_m) \cup E^0$,

(iii) $P^-h = \beta(z)z^- + \Psi(z)$,

P^- being the orthogonal projector of E onto E^-, Ψ compact, and $\beta \in C(D_m,[1,\bar{\beta}])$ where $\bar{\beta}$ depends on H.

The sets G_m are more complicated than their counterparts in Theorem 2.14 due to two factors: (a) The anologue of V in Theorem 2.14 is a subspace of the form L_m for some m and this is infinite dimensional in contrast to the earlier setting; (b) Fix $S^1 = E^0$ whereas Fix $\mathbb{Z}_2 = \{0\}$. We require $h|_{E_0} = id$ due to (b) and (iii) is needed because of (a).

Note that $id \in G_m$ for all $m \in \mathbb{N}$ so $G_m \neq \emptyset$. As earlier set

$$\Gamma_j = \{\overline{h(D_m \setminus Y)} \mid m \geq j,\ h \in G_m,\ Y \in \mathcal{E},\ i(Y) \leq m-j\}$$

where i refers to an S^1 index theory mentioned earlier such as can be found in [10] or [11]. Then the sets Γ_j possess the properties given in Proposition 2.16 and minimax values c_j can be defined as in (2.17). Moreover there is an analogue of the intersection theorem, Proposition 2.18.

Proposition 2.41: If $B \in \Gamma_j$ and $\rho < R_j$, $B \cap \partial B_\rho \cap L_{j-1}^{\perp} \neq \emptyset$. If L_{j-1} were finite dimensional, the proof of Proposition 2.18 and an S^1 version of the Borsuk-Ulam Theorem would suffice to get the result. Since L_{j-1} is infinite dimensional, a more complicated argument is required using a finite dimensional approximation argument and Property (iii) of G_m to aid in passing to a limit. See [18] for the details.

The remaining steps in the proof are to show (A) for large j, c_j is a critical value of I, (B) $c_j = I(z_j) \to \infty$ as $z_j \to \infty$ where z_j is a critical point of I corresponding to c_j, and (C) $\|z_j\|_{L^\infty} \to \infty$ as $j \to \infty$. Step (C) follows from (B). Indeed since $I'(z_j) = 0$ by (A),

$$c_j = I(z_j) = \int_0^{2\pi}(\tfrac{1}{2} z_j \cdot H_z(z_j) - H(z_j))dt \quad .$$

Thus if the functions z_j were uniformly bounded in L^∞, the numbers c_j would be bounded, contrary to (B). Steps (A) and (B) are obtained with the aid of a comparison problem. A function $M(|z|)$ is constructed which satisfies $M(|z|) > H(z)$ for all $z \in \mathbb{R}^{2n}$. Therefore

$$I(z) > J(z) \equiv \int_0^{2\pi}(p\cdot\dot{q} - M(z))dt$$

and

$$c_j > b_j \equiv \inf_{B\in\Gamma_j} \max_{u\in B} J(u) \quad .$$

E.g. if H satisfies a polynomial growth condition, $H(z) < a_1|z|^s + a_2$, we can take $M(x) = a_1x^s + a_2$. Further restrictions on the choice of M allow us to show $b_j \to \infty$ as $j \to \infty$ thus verifying (B). Lastly (A) follows for large j by a version of the Deformation Theorem for I which shows $\eta \in G_m$ for all $m \in \mathbb{N}$. For the details, consult [18].

The ideas sketched above and in fact the above theorem can be used to study the existence of periodic solutions of (1.8) on a prescribed energy surface as was mentioned in (1.10) - (1.11). As a quick example of such a result, we have

<u>Theorem 2.42</u> [19]: Suppose $H \in C^1(\mathbb{R}^{2n},\mathbb{R})$ and $S \equiv \{z \in \mathbb{R}^{2n} \mid H(z) = 1\}$ is a manifold and bounds a compact starshaped region. Then (2.38) has a periodic solution on S.

Proof: Define a new Hamiltonian $\bar{H}(z)$ as follows: Since S bounds a compact starshaped region, for any $z \in \mathbb{R}^{2n}$, $z \neq 0$, there is a unique $w(z) \in S$ and $\beta(z) > 0$ such that $z = \beta w$. It is easy to see that $\beta \in C^1$, is homogeneous of degree 1, and $\beta(z) = 1$ iff $z \in S$. Set $\bar{H}(0) = 0$ and $\bar{H}(z) = \beta(z)^4$ for $z \neq 0$. Then $\bar{H} \in C^1$ and is homogeneous of degree 4. Since S is a manifold and $S = H^{-1}(1) = \bar{H}^{-1}(1)$, there is an $a(z) \neq 0$ such that

(2.43) $$H_z(z) = a(z)\bar{H}_z(z)$$

for all $z \in S$. This fact implies any solution of (2.38) on S is a reparametrization of a solution of

(2.44) $$\dot{z} = J\,\bar{H}_z(z)$$

on S. Our above remarks about $\bar{H}$ imply it satisfies (H_1) of Theorem 2.40. Hence with e.g. $T = 2\pi$, by Theorem 2.40, (2.44) has a solution $\zeta(t)$ such that $\|\zeta(t)\|_{L^\infty} > 1$. Now ζ may not be on S. However by the homogeneity of $\bar{H}$, we can choose $\delta > 0$ such that $\bar{H}(\delta\zeta) = 1$. Moreover $\delta\zeta$ satisfies

(2.45) $$\delta\dot{\zeta} = \delta\, J\bar{H}_z(\zeta) = \delta^{-2}J\bar{H}_z(\delta\zeta) \quad .$$

Therefore after a change of time scale, $\delta\zeta$ will satisfy (2.44) and the proof is complete.

Remark 2.46. Using the ideas of Theorem 2.40 in a more direct fashion, one can prove the existence of multiple solutions of (2.37) on S provided that S satisfies further geometrical conditions, thereby obtaining results of Ekeland-Lasry [19] and generalizations thereof due to Berestycki-Lasry-Mancini-Ruf [20].

3. Perturbations from symmetry

In this final section some results on perturbation from symmetry will be discussed in the setting of Theorem 2.28. Consider

$$(3.1) \qquad -\Delta u = p(x,u),\ x \in \Omega;\ u = 0,\ x \in \partial\Omega$$

where Ω and p are as in Theorem 2.28. Therefore (3.1) possesses an unbounded sequence of weak solutions. Suppose (3.1) is perturbed by adding an inhomogeneous term:

$$(3.2) \qquad -\Delta u = p(x,u) + f(x),\ x \in \Omega;\ u = 0,\ x \in \partial\Omega\ .$$

The right hand side of (3.2) is no longer an odd function of u and

$$(3.3) \qquad I(u) = \int_\Omega (\tfrac{1}{2}\,|\nabla u|^2 - P(x,u) - f(x)u)dx$$

is no longer even on $E \equiv W_0^{1,2}(\Omega)$. Thus the machinery used in the proof of Theorem 2.28 does not apply directly to this situation.

The perturbed problem (3.2) was first treated independently by Bahri and Berestycki [21] and by Struve [22]. Later somewhat more general studies were made by Dong and Li [28] and this author [24]. All of these papers show (3.2) still possesses an unbounded sequence of weak solutions provided that s in (p_2) is further restricted. Recently Bahri [25] has proved under less general hypotheses but without any restriction on s beyond (p_2) that (3.2) has an unbounded sequence of weak solutions for <u>almost all</u> f (in e.g. $L^2(\Omega)$). Whether such a generic restriction is necessary remains an interesting open question.

In this section, following [24], we will outline how the ideas used in Theorems 2.14 and 2.28 together with some new tricks yield

<u>Theorem 3.4</u>: Suppose $f \in L^2(\Omega)$ and p satisfies $(p_1) - (p_4)$ with s further restricted by

$$(3.5) \qquad \beta \equiv \frac{(n+2)-(n-2)s}{n(s-1)} > \frac{\mu}{\mu-1}\ .$$

Then (3.2) possesses an unbounded sequence of weak solutions.

The solutions of (3.2) will be obtained as critical points of I as defined in (3.3). However there is a technical problem in working directly with I since our argument requires an estimate on its deviation from symmetry (i.e. $I(u) - I(-u)$) that I itself does not satisfy. Therefore a modified functional J will be introduced for

which the appropriate estimate can be obtained and large critical values of which are also critical values of I.

To motivate the modified problem, a priori bounds for critical points of I will be obtained in terms of the corresponding critical value. Note that by (p_3) there are constants $a_3, a_4, a_5 > 0$ such that

$$\frac{1}{\mu}(\xi\, p(x,\xi) + a_3) \geq P(x,\xi) + a_4 \geq a_5|\xi|^\mu \tag{3.6}$$

for all $\xi \in \mathbb{R}$. If u is a critical point of I, by (3.6)

$$\begin{aligned} I(u) = I(u) - \frac{1}{2} I'(u)u &\geq (\frac{1}{2} - \frac{1}{\mu})\int_\Omega (up(x,u) + a_3)dx \\ &\quad - \frac{1}{2}\|f\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} - a_6 \ . \end{aligned} \tag{3.7}$$

Using (3.6) again and the Hölder and Young inequalities, (3.7) easily leads to the a priori bound:

$$\int_\Omega (P(x,u) + a_4)dx \leq a_7(I(u)^2 + 1)^{1/2} \tag{3.8}$$

for a critical point in terms of the corresponding critical value. A bound for $\|u\|$ in terms of $I(u)$ now can be obtained from (3.8), (3.6), and the weak form of (3.1) but (3.8) suffices for our later purposes.

A modified functional can now be defined as follows: Choose $\chi \in C^\infty(\mathbb{R},\mathbb{R})$ such that $\chi(t) = 1$ for $t \leq 1$, $\chi(t) = 0$ for $t \geq 2$, and $-2 < \chi' < 0$ for $t \in (1,2)$. For $u \in E$, set $Q(u) = 2a_7(I^2(u) + 1)^{1/2}$ and

$$\psi(u) = \chi(Q(u)^{-1}\int_\Omega (P(x,u) + a_4)dx) \ .$$

Finally set

$$J(u) \equiv \int_\Omega [\frac{1}{2}|\nabla u|^2 - P(x,u) - \psi(u)f(x)u]dx \ . \tag{3.9}$$

The following proposition contains the properties of J that will be needed for what follows.

<u>Proposition 3.10</u>: If p satisfies $(p_1) - (p_4)$, $f \in L^2(\Omega)$ and ψ is as above, then

(i) $J \in C^1(E,\mathbb{R})$.

(ii) There is a constant $M > 0$ such that if $J(u) > M$ and $J'(u) = 0$, then $J(u) = I(u)$ and $I'(u) = 0$.

(iii) There is a constant $M_1 > M$ such that J satisfies (PS) relative to

$\{u \in E \mid J(u) > M_1\}$ (i.e. any sequence (u_j) such that $J(u_j) > M_1$, $J(u_j)$ is bounded from above, and $J'(u_j) \to 0$ as $j \to \infty$ is precompact).

(iv) There is a constant β_1 depending on $\|f\|_{L^2(\Omega)}$ such that for all $u \in E$,

$$|J(u) - J(-u)| \le \beta_1(|J(u)|^{\frac{1}{\mu}} + 1) \quad . \tag{3.11}$$

The proofs of these statements can be found in [24]. To prove Theorem 3.4, by (v) of Proposition 3.10, it suffices to produce an unbounded sequence of critical values of J. To do so, we begin by defining functions (v_j) via (2.32). Let $E_j = \operatorname{span}\{v_1,\ldots,v_j\}$ and $E_j^\perp$ its orthogonal complement. Replacing I by J in (2.35) and arguing as earlier shows there is an $R_j \equiv R(E_j)$ such that $J(u) < 0$ if $u \in E_j$ and $\|u\| \ge R_j$. Let $D_j = B_{R_j} \cap E_j$ and G_j be as defined in (2.15). Finally define

$$b_j = \inf_{h \in G_j} \max_{u \in D_j} J(h(u)) \quad , \quad j \in \mathbb{N} \quad . \tag{3.12}$$

These numbers cannot be expected to be critical values of I or J unless $f = 0$. However we have

<u>Proposition 3.13</u>: There exist constants $\beta_2 > 0$ and $\hat{k} \in \mathbb{N}$ (depending on $\|f\|_{L^2(\Omega)}$) such that for all $k \ge \hat{k}$,

$$b_k \ge \beta_2 \, k^{\beta} \tag{3.14}$$

where β was defined in (3.5).

<u>Proof</u>: The argument follows the same lines as (2.33) - (2.37) with the further observation [26] that

$$\lambda_j \ge \text{const.}\ j^{2/n}$$

for large j. Hence we omit the details.

The minimax values b_j will be used for comparison purposes shortly to aid in producing critical values of J. Let

$$U_j \equiv \{u = tv_{j+1} + w \mid t \in [0,R_{j+1}],\ w \in B_{R_{j+1}} \cap E_j, \text{ and } \|u\| \le R_{j+1}\}$$

and

$$\Lambda_j \equiv \{H \in C(U_j,E) \mid H|_{D_j} \in G_j \text{ and } H(u) = u \text{ if } u \in \partial B_{R_{j+1}} \cup ((B_{R_{j+1}} \setminus B_{R_j}) \cap E_j)\} \quad .$$

A new set of minimax values, c_j, can now be defined:

$$c_j = \inf_{H \in \Lambda_j} \max_{u \in U_j} J(H(u)) \qquad j \in \mathbb{N} \; . \tag{3.15}$$

<u>Proposition 3.16</u>: If $c_j > b_j \geq M_1$, $\delta \in (0, c_j - b_j)$,

$$\Lambda_j(\delta) \equiv \{H \in \Lambda_j \mid J(H(u)) \leq b_j + \delta \;\; \text{for} \;\; u \in D_j\}$$

and

$$c_j(\delta) \equiv \inf_{H \in \Lambda_j(\delta)} \max_{u \in U_j} J(H(u)) \; ,$$

then $c_j(\delta)$ is a critical value of J.

<u>Proof</u>. Since $\Lambda_j(\delta) \subset \Lambda_j$, $c_j(\delta) \geq c_j > b_j$. Suppose c_j is not a critical value of J. Let $\bar{\varepsilon} = \frac{1}{2}(c_j - b_j - \delta)$. Then by the Deformation Theorem, there is an $\eta \in C([0,1] \times E, E)$ and $\varepsilon > 0$ as earlier. In particular $\eta(1,u) = u$ if $I(u) \in [c_j(\delta) - \bar{\varepsilon}, c_j(\delta) + \bar{\varepsilon}]$. Choose $H \in \Lambda_j(\delta)$ such that

$$\max_{u \in U_j} J(H(u)) \leq c_j(\delta) + \varepsilon \; .$$

Therefore

$$\max_{u \in U_j} J(\eta(1, H(u))) \leq c_j(\delta) - \varepsilon \; . \tag{3.17}$$

We claim $\Phi \equiv \eta(1, H(\cdot)) \in \Lambda_j(\delta)$. Certainly $\Phi \in C(U_j, E)$. Moreover since $H|_{D_j} \in G_j$ and $J(H(u)) \leq b_j + \delta < c_j(\delta) - \bar{\varepsilon}$ by our choice of $\bar{\varepsilon}$, $\eta(1, H(u)) = H(u)$ for $u \in D_j$ via 1° of the Deformation Theorem. Similarly $\eta(1, H(u)) = H(u) = u$ if $u \in \partial B_{R_{j+1}} \cup$ $((B_{R_{j+1}} \; B_{R_j} \cap E_j)$. Therefore $\Phi \in \Lambda_j(\delta)$. But then

$$c_j(\delta) \leq \max_{u \in U_j} J(I(u)) \; , \tag{3.18}$$

contrary to (3.17).

<u>Completion of proof of Theorem 3.4</u>: Since $b_j \to \infty$ as $j \to \infty$, by Proposition 3.13, if we show $c_j > b_j$ for some subsequence of j's tending to infinity, then Theorem 3.4 follows from Proposition 3.16. We will prove

<u>Proposition 3.19</u>: If $c_j = b_j$ for all $j \geq j^*$, there is a constant ω such that

$$b_j \leq \omega j^{\frac{\mu}{\mu-1}} \; . \tag{3.20}$$

Comparing (3.14) to (3.20) and recalling (3.5) then shows $c_j = b_j$ for all large j is impossible. Hence Theorem 3.4 follows

__Proof of Proposition 3.19:__ For $j > j^*$ and $\varepsilon > 0$, choose $H \in {}_j$ such that

$$\max_{u \in U_j} J(H(u)) < b_j + \varepsilon \quad . \tag{3.21}$$

The function H can be extended to D_{j+1} so that it is odd and continuous. Moreover this extension belongs to G_{j+1}. Hence

$$b_{j+1} < \max_{u \in D_{j+1}} J(H(u)) \quad . \tag{3.22}$$

But $D_{j+1} = U_j \cup (-U_j)$ and by (3.21) and (iv) of Proposition (3.10),

$$\max_{-U_j} J(H(u)) < b_j + \varepsilon + \beta_1((b_j + \varepsilon)^{\frac{1}{\mu}} + 1) \quad . \tag{3.23}$$

Combining (3.21) - (3.23) and using the fact that ε is arbitrary yields

$$b_{j+1} < b_j + \beta_1(b_j^{\frac{1}{\mu}} + 1) \tag{3.24}$$

for all $j > j^*$. A straightforward induction argument then gives (3.21) and the proof is complete.

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